TOTAL IRREGULARITY STRENGTH OF DISJOINT UNION OF CROSSED PRISM AND NECKLACE GRAPHS

P. JEYANTHI
Research Centre, Department of Mathematics
Govindammal Aditanar College for Women
Tiruchendur - 628 215,Tamil Nadu, India,
E-mail: jeyajeyanthi@rediffmail.com.

A. SUDHA
Department of Mathematics
Wavoo Wajeeha Women's College of Arts and Science
Kayalpatnam -628 204, Tamil Nadu, India,
E-mail: sudhathanalakshmi@gmail.com.

Abstract

A totally irregular total k-labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ is a labeling of vertices and edges of G in such a way that for any two different vertices x and y their vertex-weights $wt_f(x) \neq wt_f(y)$ where the vertex-weight $wt_f(x) = f(x) + \sum_{xz \in E} f(xz)$ and also for every two different edges xy and $x'y'$ of G their edge-weights $wt_f(xy) = f(x) + f(xy) + f(y)$ and $wt_f(x'y') = f(x') + f(x'y') + f(y')$ are distinct. A total irregularity strength of graph G, denoted by $ts(G)$ is defined as the minimum k for which a graph G has a totally irregular total k-labeling. In this paper, we investigate the *total irregularity strength* of the crossed prism, m copies of crossed prism, necklace and m copies of necklace.

Keywords: vertex irregular total k-labeling; edge irregular total k-labeling; total irregularity strength; n-crossed prism graph; necklace graph.
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1 Introduction

As a standard notation, assume that $G = (V, E)$ is a finite, simple and undirected graph with p vertices and q edges. A labeling of a graph is any mapping that sends some set of graph elements to a set of numbers (usually positive integers). If the domain is the vertex -set (or) the edge- set, the labeling are called vertex labeling (or) edge labeling respectively. If the domain is $V \cup E$ then we call the labeling a total labeling.

In many cases it is interesting to consider the sum of all labels associated with a graph element. This will be called the weight of an element. The graph labeling has caught the attention of many authors and many new labeling results appear every year. This popularity is not only due to the mathematical challenges of

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graph labeling, but also for the wide range of its application for instance X-ray, crystallography, coding theory, radar, astronomy, circuit, design, network design and communication design.

Chartrand et al. [8] introduced labelings of the edges of a graph G with positive integers such that the sum of the labels of edges incident with a vertex is different for all the vertices. Such labelings were called *irregular assignments* and *the irregularity strength s(G)* of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k.

Bača et al. [7] introduced an edge irregular total labeling and a vertex irregular total labeling. Ivančo and Jendroľ [10] proved that

$$tes(G) \geq max\left\{\left\lceil\frac{(|E(G)|+2)}{3}\right\rceil, \left\lceil\frac{(\Delta(G)+1)}{2}\right\rceil\right\}. \quad (1)$$

Further results on the total edge irregularity strength of the graphs are available in [1, 2, 3, 4, 9, 11, 12, 13, 14, 26, 27, 28]. Nurdin et al. [21] determined the lower bound of tvs for any connected graph G.

Theorem 1.1. *[21] Let G be a connected graph having n_i vertices of degree $i(i = \delta, \delta+1, \delta+2, \ldots, \Delta)$ where δ and Δ are the minimum and maximum degree of G, respectively. Then*

$$tvs(G) \geq max\left\{\left\lceil\frac{\delta+n_\delta}{\delta+1}\right\rceil, \left\lceil\frac{\delta+n_\delta+n_{\delta+1}}{\delta+2}\right\rceil, \ldots, \left\lceil\frac{\delta+\sum_{i=\delta}^{\Delta}(n_i)}{\Delta+1}\right\rceil\right\}. \quad (2)$$

Ahmad et al. [6] showed the lower bound of total vertex irregularity strength of any graph.

Theorem 1.2. *[6] Let G be a graph with minimum degree δ and maximum degree $\Delta(G)$, then*

$$tvs(G) \geq max_{i=0}^{\Delta(G)}\left\{\left\lceil\frac{(\sum_{p=1}^{i} n_p)+\delta(G)}{i+1}\right\rceil\right\} \quad (3)$$

where n_i represents number of vertices of degree i in G.

We found [16] the total vertex irregularity strength of corona product of some graphs. Combining the ideas of vertex irregular total k-labeling and edge irregular total k-labeling, Marzuki et al. [18] introduced the concept of *totally irregular total k-labeling*. A labeling $h : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, k\}$ to be a *totally irregular total k-labeling* of the graph G if for every two different vertices x and y the vertex-weights $wt_h(x) \neq wt_h(y)$ where the vertex-weight $wt_h(x) = h(x) + \sum_{xz \in E} h(xz)$ and also for every two different edges xy and $x'y'$ of G the edge-weights

$wt_h(xy) = h(x) + h(xy) + h(y)$ and $wt_h(x'y') = h(x') + h(x'y') + h(y')$ are distinct. The *total irregularity strength* $ts(G)$ is defined as the minimum k for which a graph G has a totally irregular total k-labeling. For the total irregularity strength of a graph G, they observed that

$$ts(G) \geq max\{tes(G), tvs(G)\}. \quad (4)$$

Further information on the total irregularity strength of graphs can be obtained from [5, 22, 23]. In [19] Meilin et al. determined the total irregularity strength of wheel graph.

$$ts(W_n) = \left\lceil \frac{2n+2}{3} \right\rceil, n \geq 3. \quad (5)$$

Rismawati Ramdani et al. [24] obtained the total irregularity strength of regular graphs and observed the following:

Theorem 1.3. *[24] Let G be an r-regular connected graph with $r \geq 1$. Let f be an optimal labeling of G such that $w_f(e) < 3tes(G)$ for every $e \in E(G)$ and $w_f(v) < (r+1)tes(G)$ for every $v \in V(G)$. Then, $ts(mG) \leq m(ts(G)-1)+1$.*

In [15] we found the total irregularity strength of wheel related graphs.

2 Main Results

In this section,we determine the total irregularity strength of the n-crossed prism, m copies of crossed prism,necklace and m copies of necklace graph. Further we show that these graphs admit totally irregular total k-labeling. In addition we determine the exact value of their total irregularity strength.

Definition 2.1. *A n-crossed prism graphs for positive even n is a graph obtained by taking two disjoint cycle graphs C_n and adding edges $\{a_ib_{i+1} : 1 \leq i \leq n, i \equiv 0(mod2)\}$ and $\{b_i, a_{i+1} : 2 \leq i \leq n, i \equiv 0(mod2)\}$ and a_1b_n. The n-crossed prism graph is isomorphic to the Haar graph $H(2^{n+1} + 2^{\frac{n}{2}} + 1)$.*

Theorem 2.2. *For $n \geq 4$ and is even,then the total irregularity strength of n-crossed prism graph G_n is $n+1$, that is $ts(G_n) = n+1$.*

Proof. The vertex set $V(G_n) = \{a_i, b_i, : 1 \leq i \leq n\}$ and edge set $E(G_n) = \{a_ia_{i+1}, b_ib_{i+1} : 1 \leq i \leq n\} \cup \{a_ib_{i+1} : 1 \leq i \leq n, i \equiv 0(mod2)\} \cup \{b_i, a_{i+1} : 2 \leq i \leq n, i \equiv 0(mod2)\} \cup \{a_1b_n\}$. Also $|V(G_n)| = 2n$ and $|E(G_n)| = 3n$, by (1), (2) and (4) we have $ts(G) \geq n+1$. For the reverse inequality, we define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, n+1\}$.

$$f(a_i) = n+1, 1 \leq i \leq n;$$

$$f(b_i) = \left\lceil \frac{i}{2} \right\rceil, 1 \le i \le n;$$

$$f(a_i a_{i+1}) = i, 1 \le i \le n;$$

$$f(b_i b_{i+1}) = \begin{cases} 2 + i - \lceil \frac{i}{2} \rceil - \lceil \frac{i+1}{2} \rceil, & if \ \ 1 \le i \le n-1 \\ \frac{n}{2} + 1, & if \ \ i = n; \end{cases}$$

$$f(a_i b_{i+1}) = \begin{cases} 2 + \lceil \frac{i}{2} \rceil - \lceil \frac{i+1}{2} \rceil, & if \ \ i \equiv 0(mod2), 2 \le i \le n-2 \\ 1, & if \ \ i \equiv 0(mod2), i = n; \end{cases}$$

$$f(b_i a_{i+1}) = n + 2 - i, i \equiv 0(mod2), 2 \le i \le n.$$

We observe that all the vertex and edge labels are at most $n + 1$.

The weights of the edges are

$$wt(a_i a_{i+1}) = 2n + 2 + i, 1 \le i \le n;$$

$$wt(b_i b_{i+1}) = 2 + i, 1 \le i \le n;$$

$$wt(a_i b_{i+1}) = \begin{cases} n + 3 + \lceil \frac{i}{2} \rceil, & if \ \ i \equiv 0(mod2), 2 \le i \le n-2 \\ n + 3, & if \ \ i = n; \end{cases}$$

$$wt(b_i a_{i+1}) = 2n + 3 + \left\lceil \frac{i}{2} \right\rceil - i, i \equiv 0(mod2), 2 \le i \le n.$$

Thus the weights of the edges are pair-wise distinct.

The weights of the vertices are

$$wt(a_i) = \begin{cases} n + 2i + 2 + \lceil \frac{i}{2} \rceil - \lceil \frac{i+1}{2} \rceil, & if \ \ i \equiv 0(mod2), 2 \le i \le n-2 \\ n + 2i + 1, & if \ \ i \equiv 0(mod2), i = n \\ 2n + 3 + i, & if \ \ i \equiv 1(mod2); \end{cases}$$

$$wt(b_i) = \begin{cases} n + 5 + i - \lceil \frac{i+1}{2} \rceil - \lceil \frac{i-1}{2} \rceil - \lceil \frac{i}{2} \rceil, & if \ \ i \equiv 0(mod2), 2 \le i \le n-2 \\ 4 + n + \frac{n}{2} - \lceil \frac{n-1}{2} \rceil, & if \ \ i = n \\ 4 + \frac{n}{2}, & if \ \ i = 1 \\ 5 + 2i - \lceil \frac{i+1}{2} \rceil - 2\lceil \frac{i}{2} \rceil, & if \ \ i \equiv 1(mod2); 3 \le i \le n-1. \end{cases}$$

Hence all the weights of vertices are distinct. This labeling construction shows that $ts(G_n) \le n + 1$.Combining with the lower bound, we conclude that $ts(G_n) = n + 1$. This completes the proof. Figure 1 shows a totally irregular total labeling of crossed prism G_{10}. □

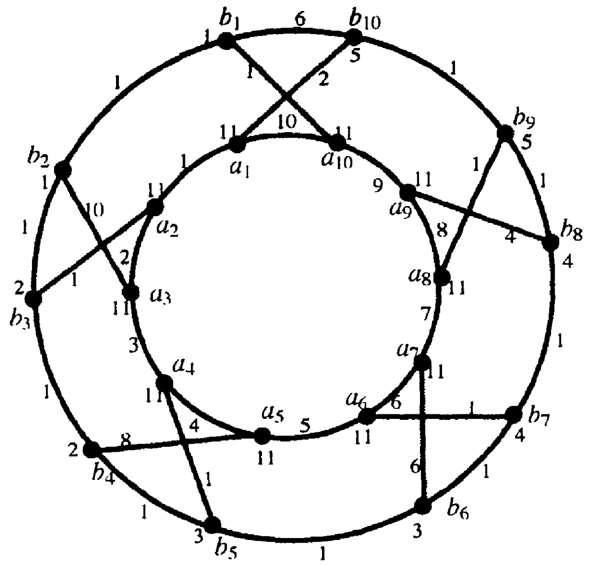

Figure 1: $ts(G_{10}) = 11$

Theorem 2.3. *Let G_n be a n-crossed prism with $2n$ vertices and $n \geq 2$. Then the total irregularity strength of disjoint union of m copies of n-crossed prism is* $mn + 1, m \geq 2$.

Proof. The graph mG_n has $2mn$ vertices and $3mn$ edges and is a 3- regular graph. From (1) and (3) we get, $tes(mG_n) \geq \lceil \frac{3mn+2}{3} \rceil$ and $tvs(mG_n) \geq \lceil \frac{2nm+3}{4} \rceil$. Therefore from (4) we get

$$ts(mG_n) \geq \left\lceil \frac{3nm+2}{3} \right\rceil = mn + 1. \tag{6}$$

By Theorem (2.2), there are no two vertices and edges has the same weight. Moreover, $w_f(e) < 3tes(G_n)$ for every $e \in E(G_n)$ and $w_f(v) < 4tes(G_n)$ for every $v \in V(G_n)$. Hence by Theorem (1.3), we get

$$\begin{aligned} ts(mG_n) \leq m(ts(G) - 1) + 1 &= m(n + 1 - 1) + 1 \\ &= mn + 1. \end{aligned}$$

This implies that,

$$ts(mG_n) \leq mn + 1. \tag{7}$$

From (6) and (7) we get

$$ts(mG_n) = mn + 1.$$

□

Definition 2.4. *Let P be a path of length $n - 1$ with vertices labeled from a_1*

to a_n along P. The comb Cb_n is the tree consisting of P together with vertices $b_1, b_2, b_3, \dots, b_{n-2}$ and edges b_1a_2 through $b_{n-2}a_{n-1}$. The necklace graph Ne_n is obtained from Cb_n by the adding the edges $a_1b_1, a_1a_n, b_1b_2, \dots, b_{n-3}b_{n-2}$ and $b_{n-2}a_n$. The vertices a_1 and a_n are called end vertices of the Ne_n. The necklace graph Ne_3 is isomorphic to W_3.

Lemma 2.5. *The total irregularity strength of m copies of necklace graph Ne_3 is $2m+1$, that is $ts(mNe_3) = 2m+1, m \geq 2$. This graph Ne_3 is called wheel graph W_3.*

Proof. The graph mNe_3 has $4m$ vertices and $6m$ edges and is a 3- regular graph. From (1) and (3) we get, $tes(mNe_3) \geq \left\lceil \frac{6m+2}{3} \right\rceil$ and $tvs(mNe_3) \geq \left\lceil \frac{4m+3}{4} \right\rceil$.
Therefore from (4) we get,

$$ts(mNe_3) \geq \left\lceil \frac{6m+2}{3} \right\rceil = 2m+1. \tag{8}$$

By (5),

$$ts(W_n) = \left\lceil \frac{2n+2}{3} \right\rceil, n \geq 3.$$

Therefore $ts(W_3) = ts(Ne_3) = 3$.

The labeling for Ne_3 is

$$f(a_1) = 1; f(a_2) = 1; f(a_3) = 3; f(b_1) = 2; f(a_1a_2) = 1;$$
$$f(a_2a_3) = 3; f(b_1a_1) = 1; f(b_1a_2) = 2; f(b_1a_3) = 3.$$

The weights of the edges and vertices are

$wt(a_1a_2) = 3; wt(a_2a_3) = 7; wt(b_1a_1) = 4; wt(b_1a_2) = 5; wt(b_1a_3) = 8;$
$wt(a_1) = 5; wt(a_2) = 7; wt(a_3) = 11; wt(b_1) = 8.$

We observe that, the weights of the edges and vertices receive distinct values. Moreover, $w_f(e) < 9$ for every $e \in E(Ne_3)$ and $w_f(v) < 12$ for every $v \in V(Ne_3)$. Hence by Theorem (1.3), we get

$$ts(mNe_3) \leq m(ts(Ne_3) - 1) + 1 = m(3-1) + 1$$
$$= 2m+1.$$

This implies that,

$$ts(mNe_3) \leq 2m+1. \tag{9}$$

From (8) and (9) we get,

$$ts(mNe_3) = ts(mW_3) = 2m + 1.$$

Lemma 2.6. *The total irregularity strength of m copies of necklace graph Ne_4 is $3m + 1$, that is $ts(mNe_4) = 3m + 1, m \geq 2$ and this graph is called Halin graph $H(S_{3,3})$.*

Proof. In the necklace graph Ne_4, $|V(Ne_4)| = 6$ and $|E(Ne_4)| = 9$. From (1), (2) and (3) we have

$$ts(Ne_4) \geq 4. \tag{10}$$

The labeling for Ne_4 is

$f(a_1) = 1; f(a_2) = 2; f(a_3) = 4; f(a_4) = 4; f(b_1) = 1; f(b_2) = 4; f(a_1a_2) = 2;$

$f(a_2a_3) = 2; f(a_3a_4) = 3; f(b_1a_1) = 1; f(b_1a_2) = 1; f(b_2a_3) = 2; f(b_2a_4) = 1.$

The weights of the edges and vertices are

$wt(b_1a_1) = 3; wt(b_1a_2) = 4; wt(b_2a_3) = 10; wt(b_2a_4) = 9; wt(a_1a_2) = 5; wt(a_2a_3) = 8; wt(a_3a_4) = 11; wt(a_1) = 6; wt(a_2) = 7; wt(a_3) = 11; wt(a_4) = 10; wt(b_1) = 4; wt(b_1) = 8.$
we observe that, the weights of the edges and vertices receive distinct values. This labeling construction shows that

$$ts(Ne_4) \leq 4. \tag{11}$$

From(10) and(11) we get

$$ts(Ne_4) = 4. \tag{12}$$

The graph mNe_4 has $6m$ vertices and $9m$ edges and is a 3-regular graph.
From (1) and (3) we get, $tes(mNe_4) \geq \left\lceil \frac{9m+2}{3} \right\rceil$ and $tvs(mNe_4) \geq \left\lceil \frac{6m+3}{4} \right\rceil$.
Therefore from (4) we get

$$ts(mNe_4) \geq \left\lceil \frac{9m+2}{3} \right\rceil = 3m + 1. \tag{13}$$

We observe that, the weights of vertices and edges of Ne_4 are distinct. Moreover, $w_f(e) < 12$ for every $e \in E(Ne_4)$ and $w_f(v) < 16$ for every $v \in V(Ne_4)$. Hence by Theorem (1.3), we get

$$ts(mNe_4) \leq m(ts(Ne_4) - 1) + 1 = m(4 - 1) + 1$$

$$= 3m + 1.$$

This implies that,

$$ts(mNe_4) \leq 3m + 1. \quad (14)$$

From (13) and (14) we get,

$$ts(mNe_4) = ts(mH(S_{3,3})) = 3m + 1.$$

Theorem 2.7. *For $n \geq 5$, then the total irregularity strength of necklace graph Ne_n is n, that is $ts(Ne_n) = n$.*

Proof. The vertex set $V(Ne_n) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n-2\}$ and edge set $E(Ne_n) = \{a_i a_{i+1} : 1 \leq i \leq n\} \cup \{b_i b_{i+1} : 1 \leq i \leq n\} \cup \{a_1 b_1; b_{n-2} a_n\} \cup \{b_i a_{i+1} : 1 \leq i \leq n-2\}$. Also $|V(Ne_n)| = 2n-2$ and $|E(Ne_n)| = 3n-3$, by (1), (2) and (4) we have $ts(Ne_n) \geq n$. For the reverse inequality, we define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, n\}$ by considering the following two cases:

Case (1): n is even

$$f(a_i) = \begin{cases} n, & if\ i = 1 \\ i-1, & if\ 2 \leq i \leq n-2 \\ i, & if\ i = n-1, n; \end{cases}$$

$$f(b_i) = i, 1 \leq i \leq n;$$

$$f(a_1 b_1) = n - 1;$$

$$f(a_i a_{i+1}) = \begin{cases} n, & if\ i = 1 \\ i, & if\ 2 \leq i \leq \left\lfloor \frac{2n}{3} \right\rfloor \\ i+2, & if\ \left\lfloor \frac{2n}{3} \right\rfloor + 1 \leq i \leq n-3 \\ i+1, & if\ i = n-2 \\ i-1, & if\ i = n-1 \\ n, & if\ i = n; \end{cases}$$

$$f(b_i a_{i+1}) = \begin{cases} i, & if\ 1 \leq i \leq \left\lceil \frac{2n}{3} \right\rceil - 1 \\ i+2, & if\ \left\lceil \frac{2n}{3} \right\rceil \leq i \leq n-3 \\ n-1, & if\ i = n-2; \end{cases}$$

$$f(b_{n-2} a_n) = n;$$

$$f(b_i b_{i+1}) = \begin{cases} i, & if\ 1 \leq i \leq \left\lceil \frac{2n}{3} \right\rceil - 1 \\ i+2, & if\ \left\lceil \frac{2n}{3} \right\rceil \leq i \leq n-3. \end{cases}$$

We observe that all the vertex and edge labels are at most n and the weights of the edges and vertices are as follows:

$$wt(a_i a_{i+1}) = \begin{cases} 2n+1, & if\ i = 1 \\ 3i-1, & if\ 2 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 3i+1, & if\ \left\lfloor \frac{2n}{3} \right\rfloor + 1 \le i \le n-2 \\ 3n-3, & if\ i = n-1 \\ 3n, & if\ i = n; \end{cases}$$

$$wt(b_i b_{i+1}) = \begin{cases} 3i+1, & if\ 1 \le i \le \left\lceil \frac{2n}{3} \right\rceil - 1 \\ 3i+3, & if\ \left\lceil \frac{2n}{3} \right\rceil \le i \le n-3; \end{cases}$$

$$wt(b_i a_{i+1}) = \begin{cases} 3i, & if\ 1 \le i \le \left\lceil \frac{2n}{3} \right\rceil - 1 \\ 3i+2, & if\ \left\lceil \frac{2n}{3} \right\rceil \le i \le n-3 \\ 3n-4, & if\ i = n-2; \end{cases}$$

$$wt(b_{n-2} a_n) = 3n-2;$$

$$wt(b_1 a_1) = 2n.$$

Thus the weights of edges are pair-wise distinct.
The weights of vertices are

$$wt(a_i) = \begin{cases} 4n-1, & if\ i = 1 \\ n+4, & if\ i = 2 \\ 4i-3, & if\ 3 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 4i+1, & if\ n \equiv 0(mod3), i = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \\ 4i-1, & if\ n \equiv 1, 2(mod3), i = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \\ 4i+3, & if\ \left\lfloor \frac{2n}{3} \right\rfloor + 2 \le i \le n-3 \\ 4n-6, & if\ i = n-2 \\ 4n-5, & if\ i = n-1 \\ 4n-2, & if\ i = n; \end{cases}$$

The weights of b_i has the following three subcases:

Subcase (i) : $n \equiv 0(mod3)$

$$wt(b_i) = \begin{cases} n+2, & if \;\; i = 1 \\ 4i-1, & if \;\; 2 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor - 1 \\ 4i+3, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor \\ 4i+5, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \le i \le n-3 \\ 4n-4, & if \;\; i = n-2. \end{cases}$$

Subcase (ii) : $n \equiv 1(mod3)$

$$wt(b_i) = \begin{cases} n+2, & if \;\; i = 1 \\ 4i-1, & if \;\; 2 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor \\ 4i+3, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \\ 4i+5, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor + 2 \le i \le n-3 \\ 4n-4, & if \;\; i = n-2. \end{cases}$$

Subcase (iii) : $n \equiv 2(mod3)$

$$wt(b_i) = \begin{cases} n+2, & if \;\; i = 1 \\ 4i-1, & if \;\; 2 \le i \le \left\lfloor \frac{2n}{3} \right\rfloor - 1 \\ 4i+1, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor \\ 4i+5, & if \;\; i = \left\lfloor \frac{2n}{3} \right\rfloor + 1 \le i \le n-3 \\ 4n-4, & if \;\; i = n-2. \end{cases}$$

Hence the weights of the vertices are distinct.

Case (2): n is odd

Subcase (i): $n \equiv 0(mod3)$

$$f(a_1) = 1;$$

$$f(a_i) = \begin{cases} i-1, & if \;\; 2 \le i \le n-2 \\ i, & if \;\; i = n-1, n; \end{cases}$$

$$f(bi) = \begin{cases} i, & if \;\; 1 \le i \le n-3 \\ n-1, & if \;\; i = n-2; \end{cases}$$

$$f(a_1a_2) = 1; f(a_1a_n) = 1; f(b_{n-2}a_{n-1}) = n-1;$$

$$f(b_{n-2}a_n) = n; f(b_1a_1) = 2; f(b_1a_2) = 3;$$

$$f(a_ia_{i+1}) = \begin{cases} i+1, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil \\ i+2, & if\ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le n-3 \\ n-1, & if\ i = n-2 \\ n-1, & if\ i = n-1; \end{cases}$$

$$f(b_ib_{i+1}) = \begin{cases} i+3, & if\ 1 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ i+4, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-4 \\ n, & if\ i = n-3; \end{cases}$$

$$f(b_ia_{i+1}) = \begin{cases} i+2, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ i+3, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-3. \end{cases}$$

We observe that all the vertex and edge labels are at most n. Then the weights of the edges and vertices are as follows:

$$wt(a_ia_{i+1}) = \begin{cases} 3, & if\ i = 1 \\ 6, & if\ i = 2 \\ 3i, & if\ 3 \le i \le \left\lceil \frac{n}{3} \right\rceil \\ 3i+1, & if\ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le n-1 \\ n+2, & if\ i = n; \end{cases}$$

$$wt(b_ib_{i+1}) = \begin{cases} 7, & if\ i = 1 \\ 3i+4, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ 3i+5, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-3; \end{cases}$$

$$wt(b_ia_{i+1}) = \begin{cases} 5, & if\ i = 1 \\ 3i+2, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ 3i+3, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-3 \\ 3n-3, & if\ i = n-2; \end{cases}$$

$$wt(b_{n-2}a_n) = 3n-1; wt(b_1a_1) = 4.$$

Thus the weights of the edges are pair-wise distinct. The weights of the vertices are

$$wt(a_i) = \begin{cases} 5, & if\ i = 1 \\ 8, & if\ i = 2 \\ 4i + 1, & if\ 3 \le i \le \lceil \frac{n}{3} \rceil \\ 4i + 3, & if\ i = \lceil \frac{n}{3} \rceil + 1 \\ 4i + 4, & if\ \lceil \frac{n}{3} \rceil + 2 \le i \le n - 3 \\ 4n - 5, & if\ i = n - 2 \\ 4n - 4, & if\ i = n - 1 \\ 3n, & if\ i = n; \end{cases}$$

$$wt(b_i) = \begin{cases} 10, & if\ i = 1 \\ 4i + 7, & if\ 2 \le i \le \lceil \frac{n}{3} \rceil - 1 \\ 4i + 9, & if\ i = \lceil \frac{n}{3} \rceil \\ 4i + 10, & if\ \lceil \frac{n}{3} \rceil + 1 \le i \le n - 4 \\ 4n - 3, & if\ i = n - 3 \\ 4n - 2, & if\ i = n - 2. \end{cases}$$

Hence the weights of the edges and vertices receive distinct values.

Subcase (ii) : $n \equiv 1 (mod 3)$

$$f(a_1) = 1;$$

$$f(a_i) = \begin{cases} i - 1, & if\ 2 \le i \le n - 2 \\ i, & if\ i = n - 1, n; \end{cases}$$

$$f(bi) = \begin{cases} i, & if\ 1 \le i \le n - 3 \\ n - 1, & if\ i = n - 2; \end{cases}$$

$$f(a_1 a_2) = 1; f(a_1 a_n) = 1; f(b_{n-2} a_{n-1}) = n - 1;$$

$$f(b_{n-2} a_n) = n; f(b_1 a_1) = 2; f(b_1 a_2) = 3;$$

$$f(a_i a_{i+1}) = \begin{cases} i + 1, & if\ 2 \le i \le \lceil \frac{n}{3} \rceil - 1 \\ i + 2, & if\ \lceil \frac{n}{3} \rceil \le i \le n - 3 \\ n - 1, & if\ i = n - 2 \\ n - 1, & if\ i = n - 1; \end{cases}$$

$$f(b_i b_{i+1}) = \begin{cases} i + 3, & if\ 1 \le i \le \lceil \frac{n}{3} \rceil - 2 \\ i + 4, & if\ \lceil \frac{n}{3} \rceil - 1 \le i \le n - 4 \\ n, & if\ i = n - 3; \end{cases}$$

$$f(b_i a_{i+1}) = \begin{cases} i+2, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ i+3, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-3. \end{cases}$$

We observe that all the vertex and edge labels are at most n. Then the weights of the edges and vertices are as follows:

$$wt(a_i a_{i+1}) = \begin{cases} 3, & if\ i = 1 \\ 6, & if\ i = 2 \\ 3i, & if\ 3 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ 3i+1, & if\ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le n-1 \\ n+2, & if\ i = n; \end{cases}$$

$$wt(b_i b_{i+1}) = \begin{cases} 7, & if\ i = 1 \\ 3i+4, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 2 \\ 3i+5, & if\ \left\lceil \frac{n}{3} \right\rceil - 1 \le i \le n-3; \end{cases}$$

$$wt(b_i a_{i+1}) = \begin{cases} 5, & if\ i = 1 \\ 3i+2, & if\ 2 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ 3i+3, & if\ \left\lceil \frac{n}{3} \right\rceil \le i \le n-3; \\ 3n-3, & if\ i = n-2; \end{cases}$$

$$wt(b_{n-2} a_n) = 3n - 1; wt(b_1 a_1) = 4.$$

Thus the weights of edges are pair-wise distinct. The weights of vertices are

$$wt(a_i) = \begin{cases} 5, & if\ i = 1 \\ 8, & if\ i = 2 \\ 4i+1, & if\ 3 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \\ 4i+2, & if\ i = \left\lceil \frac{n}{3} \right\rceil \\ 4i+4, & if\ \left\lceil \frac{n}{3} \right\rceil + 1 \le i \le n-3 \\ 4n-5, & if\ i = n-2 \\ 4n-4, & if\ i = n-1 \\ 3n, & if\ i = n; \end{cases}$$

$$wt(b_i) = \begin{cases} 10, & if \ i = 1 \\ 4i + 7, & if \ 2 \le i \le \lceil \frac{n}{3} \rceil - 2 \\ 4i + 8, & if \ i = \lceil \frac{n}{3} \rceil - 1 \\ 4i + 10, & if \ \lceil \frac{n}{3} \rceil \le i \le n - 4 \\ 4n - 3, & if \ i = n - 3 \\ 4n - 2, & if \ i = n - 2. \end{cases}$$

Hence the weights of vertices are distinct.

Subcase (iii) : $n \equiv 2(mod 3)$

$$f(a_1) = 1;$$

$$f(a_i) = \begin{cases} i - 1, & if \ 2 \le i \le n - 2 \\ i, & if \ i = n - 1, n; \end{cases}$$

$$f(bi) = \begin{cases} i, & if \ 1 \le i \le n - 3 \\ n - 1, & if \ i = n - 2; \end{cases}$$

$$f(a_1a_2) = 1; f(a_1a_n) = 1; f(b_{n-2}a_{n-1}) = n - 1;$$

$$f(b_{n-2}a_n) = n; f(b_1a_1) = 2; f(b_1a_2) = 3;$$

$$f(a_ia_{i+1}) = \begin{cases} i + 1, & if \ 2 \le i \le \lceil \frac{n}{3} \rceil \\ i + 2, & if \ \lceil \frac{n}{3} \rceil + 1 \le i \le n - 3 \\ n - 1, & if \ i = n - 2 \\ n - 1, & if \ i = n - 1; \end{cases}$$

$$f(b_ib_{i+1}) = \begin{cases} i + 3, & if \ 1 \le i \le \lceil \frac{n}{3} \rceil - 2 \\ i + 4, & if \ \lceil \frac{n}{3} \rceil - 1 \le i \le n - 4 \\ n, & if \ i = n - 3; \end{cases}$$

$$f(b_ia_{i+1}) = \begin{cases} i + 2, & if \ 2 \le i \le \lceil \frac{n}{3} \rceil - 1 \\ i + 3, & if \ \lceil \frac{n}{3} \rceil \le i \le n - 3. \end{cases}$$

We observe that all the vertex and edge labels are at most n. Then the weights of the edges and vertices are as follows:

$$wt(a_ia_{i+1}) = \begin{cases} 3, & if\ i = 1 \\ 6, & if\ i = 2 \\ 3i, & if\ 3 \le i \le \lceil \frac{n}{3} \rceil \\ 3i+1, & if\ \lceil \frac{n}{3} \rceil + 1 \le i \le n-1 \\ n+2, & if\ i = n; \end{cases}$$

$$wt(b_ib_{i+1}) = \begin{cases} 7, & if\ i = 1 \\ 3i+4, & if\ 2 \le i \le \lceil \frac{n}{3} \rceil - 2 \\ 3i+5, & if\ \lceil \frac{n}{3} \rceil - 1 \le i \le n-3; \end{cases}$$

$$wt(b_ia_{i+1}) = \begin{cases} 5, & if\ i = 1 \\ 3i+2, & if\ 2 \le i \le \lceil \frac{n}{3} \rceil - 1 \\ 3i+3, & if\ \lceil \frac{n}{3} \rceil \le i \le n-3; \\ 3n-3, & if\ i = n-2; \end{cases}$$

$$wt(b_{n-2}a_n) = 3n-1; wt(b_1a_1) = 4.$$

Thus the weights of edges are pair-wise distinct. The weights of vertices are

$$wt(a_i) = \begin{cases} 5, & if\ i = 1 \\ 8, & if\ i = 2 \\ 4i+1, & if\ 3 \le i \le \lceil \frac{n}{3} \rceil \\ 4i+3, & if\ i = \lceil \frac{n}{3} \rceil + 1 \\ 4i+4, & if\ \lceil \frac{n}{3} \rceil + 2 \le i \le n-3 \\ 4n-5, & if\ i = n-2 \\ 4n-4, & if\ i = n-1 \\ 3n, & if\ i = n; \end{cases}$$

$$wt(b_i) = \begin{cases} 10, & if\ i = 1 \\ 4i+7, & if\ 2 \le i \le \lceil \frac{n}{3} \rceil - 2 \\ 4i+8, & if\ i = \lceil \frac{n}{3} \rceil - 1 \\ 4i+10, & if\ \lceil \frac{n}{3} \rceil \le i \le n-4 \\ 4n-3, & if\ i = n-3 \\ 4n-2, & if\ i = n-2. \end{cases}$$

Hence the weights of edges and vertices are distinct. From Cases (1) and (2), this labeling construction shows that $ts(Ne_n) \le n$. Combining with the lower bound, we conclude that, $ts(Ne_n) = n$. □

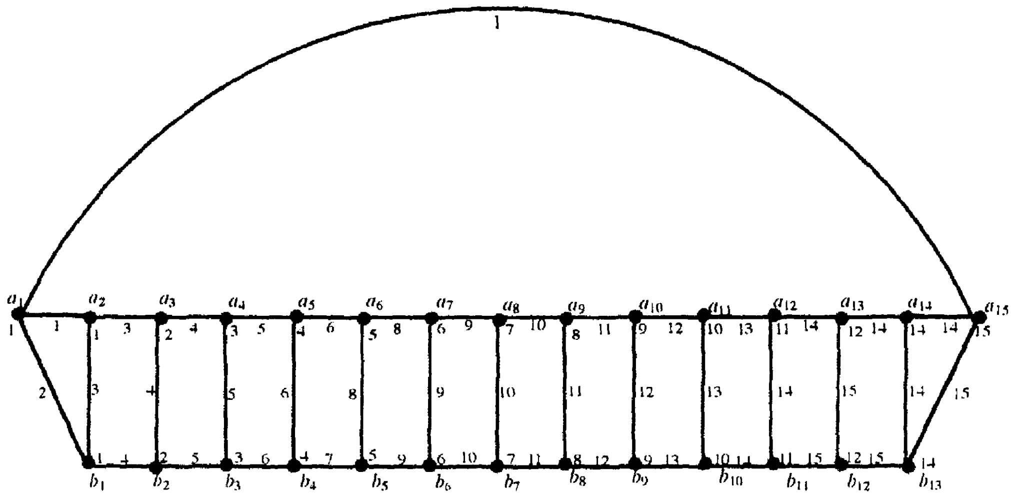

Figure 2: $ts(Ne_{15}) = 15$.

Theorem 2.8. *For $n \geq 3$ and is odd, Let Ne_n be a necklace graph with $2n - 2$ vertices. Then the total irregularity strength of disjoint union of $m \geq 2$ copies of necklace graph is $mn - m + 1$.*

Proof. The graph Ne_n has $2mn-2m$ vertices and $3mn-3m$ edges and is a 3-regular graph. From (1) and (3) we get, $tes(Ne_n) \geq \left\lceil \frac{3(mn-m)+2}{3} \right\rceil$ and $tvs(Ne_n) \geq \left\lceil \frac{2nm-2m+3}{4} \right\rceil$

Therefore from (4) we get

$$ts(mNe_n) \geq \left\lceil \frac{3(nm-m)+2}{3} \right\rceil = mn - m + 1. \tag{15}$$

By Theorem (2.7), there are no two vertices and edges has the same weight. Moreover, $w_f(e) < 3tes(Ne_n)$ for every $e \in E(Ne_n)$ and $w_f(v) < 4tes(Ne_n)$ for every $v \in V(Ne_n)$. Also, from Theorem (1.3), we get

$$ts(mNe_n) \leq m(ts(Ne_n) - 1) + 1 = m(n-1) + 1 = mn - m + 1.$$

This implies that,

$$ts(mNe_n) \leq mn - m + 1. \tag{16}$$

From (15) and (16) we get

$$ts(mNe_n) = mn - m + 1.$$

□

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